Homotopy groups of the complements to singular hypersurfaces, II

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Abstract. The homotopy group $\pi_{n-k}(\mathbf{C}^{n+1} - V)$ where V is a hypersurface with a singular locus of dimension k and good behavior at infinity is described using generic pencils. This is analogous to the van Kampen procedure for finding a fundamental group of a plane curve. In addition we use a certain representation generalizing the Burau representation of the braid group. A divisibility theorem is proven that shows the dependence of this homotopy group on the local type of singularities and behavior at infinity. Examples are given showing that this group depends on certain global data in addition to local data on singularities.

0. Introduction.

The fundamental groups of complements to algebraic curves in $\mathbf{CP}^2$ were studied by O.Zariski almost 60 years ago (cf. [Z]). He showed that these groups are affected by the type and position of singularities. Zariski and Van Kampen (cf.[vK]) described a general procedure for calculating these groups in terms of the behavior of the intersection of the

* Supported by NSF grant

curve with a generic line while one varies this line in a pencil. For the curves with mild and few singularities, these fundamental groups are abelian. For example, if C is an irreducible curve, having only singular points near which C can be given in some coordinate system by the equation $x^2 = y^2$, then its complement has an abelian fundamental group (cf. [F],[De]). On the other hand, one knows that there is an abundance of curves with non-abelian fundamental groups of complements (for example, branching curves of generic projections on a plane of surfaces embedded in some projective space). Some explicit calcualtions were made by O.Zariski. For example, for the curve given by equation $f_2^3 + f_3^2 = 0$, where $f_k(z_0, z_1, z_2)$ is a generic form of degre k, the corresponding fundamental group is $PSL_2(\mathbf{Z})$ (cf [Z]).

If one thinks of the high dimensional analog of these results one can immediately notice that the class of fundamental groups of the complements to hypersurfaces in a projective space coincides with the class of fundamental groups of complements to the curves. Indeed by a Zariski-Lefschetz type theorem (cf. [H]) the fundamental group of the complement to a hypersurface V is the same as the fundamental group of the complement to the intersection $V \cap H$ inside H for generic plane H. In this paper we will show, however, that the homotopy group $\pi_{n-k}(\mathbf{C}^{n+1} - V)$, where k is the dimension of the singular locus of V, exhibits properties rather similar to the properties of the fundamental group discovered by O.Zariski. (By abuse of notation we often will omit specifying the base point in homotopy groups except when it can cause confusion). Actually, as in the case of curves (cf. [Z],[vK]), we study a somewhat more general case of hypersurfaces in affine space (which is motivated by a desire to apply these results to the covers of $\mathbf{CP}^{n+1}$ of arbitrary degree branching locus of which contains V). It appears that the group $\pi_{n-k}(\mathbf{CP}^{n+1} - (V \cup H))$ where H is "a hyperplane at infinity", at least after tensoring with $\mathbf{Q}$, has a geometric (rather than a homotopy theoretic) meaning: it depends on the "local type and position" of singularities of section of V by a generic linear subspace of $\mathbf{C}^{n+1}$ of codimension k (cf. example 5.4). This group has a description in the spirit of geometric topology similar to the one given

by the van Kampen theorem in which the Artin braid group is replaced by a certain generalization. (cf.sect.2). We allow V to have a certain type of singular behavior at infinity. The information on the homotopy groups which we obtain in the affine situation when V is non-singular at infinity, as we show below, is equivalent to the information on the homotopy groups of complements to projective hypersurfaces. An a priori construction like the one with generic projections mentioned above in the case of fundamental groups seems absent in our context (though the case of weighted homogeneous hypersurfaces discussed in section 1 provides class of hypersurfaces with one singular point in $\mathbf{C}^{n+1}$ and non-trivial π_n). However, one can, starting from the equation of curves with non-trivial fundamental groups of the complements, construct equations of hypersurfaces having a non-trivial higher homotopy of their complements. In particular, one obtains hypersurfaces with the same local data but with a distinct higher homotopy of the complement.

A more detailed content of this paper is the following. In section 1 we start with a study of the complements to non-singular hypersurfaces in $\mathbf{C}^{n+1}$ that, in particular, implies that the dimension $n-k$, where k as above is the dimension of the singular locus of V, is the lowest in which non-trivial homotopy groups can appear. It also allows one to reduce the study of π_{n-k} to the case when V has only isolated singularities. Moreover, we also derive a relation between $\pi_n(\mathbf{CP}^{n+1}-V)$ and $\pi_n(\mathbf{C}^{n+1}-V)$ where V is a hypersurface with isolated singularities and no singularities at infinity. In section 2 we outline a procedure for finding $\pi_n(\mathbf{C}^{n+1}-V)$ using a generic pencil of hyperplane sections of V. Such a pencil defines the geometric monodromy homomorphism of the fundamental group of the space of parameters of the pencil corresponding to non singular members of this pencil. The target of this homomorphism is the fundamental group of the space of certain embeddings of $V \cap H$ into $H = \mathbf{C}^n$ where H is a generic element of the pencil. The latter group has a natural linear representaion over $\mathbf{Z}[t, t^{-1}]$. Then the homotopy group in question is expressed in terms of the geometric monodromy and this linear representation (in some cases information on certain degeneration operators will be needed). In the case of curves, the whole procedure

coincides with van Kampen's (cf. [vK]), the group of embeddings is the Artin braid group and the geometric monodromy is the braid monodromy (cf. [Mo]). The theorem 2.4 reduces then to van Kampen's theorem. The linear representation mentioned above is the classical Burau representation and the object which the theorem 2.4 calculates is the Alexander module of the curve (cf. [L2]). Section 3 describes a neccesary condition for the vanishing of $\pi_n(\mathbf{C}^{n+1} - V)$. Use of this result shows that in some cases the contribution from the degeneration operators can be omitted. The vanishing results here are parallel to the results on the commutativity of the fundamental groups of complements in the case of curves. The issue of explicit numerical conditions on singularities, which will assure the vanishing of the homotopy groups, is more algebro-geometric in nature than most issues treated here and will be discussed elsewhere. Note also that another vanishing result for π_2 of the complement to an image of a generic projection was obtained in [L3]. The next section gives restrictions on $\pi_n(\mathbf{C}^{n+1} - V)$ imposed by the local type of singularities and the behavior of V at infinity. These results are generalizations of divisibility theorems for Alexander polynomials in [L2]. As a corollary, we show that in the case of the absence of singularities at infinity the order of $\pi_n(\mathbf{C}^{n+1} - V) \otimes \mathbf{Q}$ coincides with the characteristic polynomial of the monodromy operator acting on H_n of the Milnor fibre of (non-isolated) singularity of the defining equation of V at the origin in $\mathbf{C}^{n+2}$. These characteristic polynomials were also considered in [Di]. In the last section we give two methods for constructing hypersurfaces for which $\pi_n(\mathbf{C}^{n+1} - V) \neq 0$ and we calculate the homotopy groups in these cases. The first method is based on a generalization of Zariski's example mentioned in the first paragraph . We show that if f_k denotes a generic form of degree k of $n+2$ variables, p_i $i = 1, .., n+1$ are positive integers and $q_i = (\prod_{i=1}^{n+1} p_i)/p_i$, then the order $\pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}$ (H is a generic hyperplane) as a module over $\mathbf{Q}[t, t^{-1}]$ (cf. sect.1) where V is given by equation

$$f_{q_1}^{p_1} + ... f_{q_{n+1}}^{p_{n+1}} = 0 \qquad (0.1)$$

is the characteristic polynomial of the monodromy of the singularity $x_1^{p_1} + ... x_{n+1}^{p_{n+1}} = 0$. The

hypersurface (0.1) and the non-vanishing of π_n was described in [L1] as a consequence of the fact that singularities of (0.1) form a finite set in $\mathbf{CP}^{n+1}$ which is a *complete intersection* of $n+1$ hypersurfaces $f_{q_1} = ...f_{q_{n+1}} = 0$ in $\mathbf{CP}^{n+1}$. The second method, based on the use of Thom-Sebastiani theorem results in examples of hypersurfaces with the same collection of singularities but with distinct π_n's of the complement.

Part of the results in this paper were outlined in the announcement [L1]. On the other, the latter contained number of results on the relationship of the homotopy groups in question with the Hodge theory of the cyclic covers of $\mathbf{CP}^{n+1}$ branched over V; this will be addressed elsewhere. Note that since [L1] appeared two more publications ([Di] and [Deg] related to the case of hypersurfaces in $\mathbf{CP}^{n+1}$ came out. Finally, I want to thank Prof.P.Deligne for discussions and the Institute for Advanced Study, where a part of this paper was written, for support and hospitality.

1 Preliminaries

This section describes the topology of complements to non-singular hypersurfaces $\mathbf{C}^{n+1}$, the case of weighted homogeneous hypersurfaces, the homology of the complements and the relationship between complements in affine and projective spaces. Note that the complements to nonsingular hypersurfaces in projective case were first studied in [KW].

Lemma 1.1. Let V be a non-singular hypersurface in $\mathbf{C}^{n+1}$ which is transversal to the hyperplane at infinity (resp. a non- singular hypersurface in $\mathbf{CP}^{n+1}$). Then $\pi_i(\mathbf{C}^{n+1}-V) = 0$ for $1 < i \leq n$ and $\pi_1(\mathbf{C}^{n+1} - V) = \mathbf{Z}$ (resp. $\mathbf{Z}/d\mathbf{Z}$, d is the degree of V).

Proof. The statement about the fundamental groups follows immediately from the Zariski theorem by taking a section of the hypersurface by generic plane. A hypersurface satisfying the conditions of the lemma is isotopic to the hypersurface given by equation:

$z_1^d + ... z_{n+1}^d = 1$. (resp. projective closure of this). The d-fold cover of $\mathbf{CP}^{n+1}$ branched over the projective closure of the latter can be identified with the hypersurface $\mathcal{V}$ in $\mathbf{C}^{n+2}$ given by $z_0^d + z_1^d + ... z_{n+1}^d = 1$. This is n-connected because, for example, it is diffeomorphic to the Milnor fibre of the isolated singularity $z_0^d + ... z_{n+1}^d = 0$. Hence our claim follows in the projective case. The d-fold cover of the complement in $\mathbf{C}^{n+1}$ is obtained by removing from $\mathcal{V}$ the hyperplane section H_0 given by $z_0 = 0$. In other words this d-fold cover is the hypersurface in $\mathbf{CP}^{n+1} - (H_0 \cap H_\infty) = \mathbf{C}^*$ (H_∞ is the hyperplane at infinity) which is transversal to these hyperplanes. Hence the Lefschetz type theorem implies that $\pi_i(\mathcal{V} - H_0) = \pi_i(\mathbf{C}^*)$.

Corollary.1.2. If V is a non-singular hypersurface transversal to the hyperplane at infinity then $\mathbf{C}^{n+1} - V$ is homotopy equivalent to the wedge of spheres $S^1 \vee S^{n+1} \vee ... \vee S^{n+1}$.

Proof. Indeed the lemma implies that CW-complex $\mathbf{C}^{n+1}$ is a $(\mathbf{Z}, n+1)$ complex in the sense of [Dy] and hence is homotopy equivalent to the wedge as above (i.e. as a consequence of stable triviality of $\pi_{n+1}(\mathbf{C}^{n+1} - V)$ (cf. [Wh],th 14) and the fact that stably trivial modules over $\mathbf{Z}[t, t^{-1}]$ are free.).

Remark 1.3. It is interesting to see how such wedge comes up geometrically. Let us compare the complement to $q(z_1, ..., z_{n+1}) = z_1^2 + ... z_{n+1}^2 = 1$ in $\mathbf{C}^{n+1}$ with the complement to $z_0^2 + ... z_{n+1}^2 = 0$. The last complement fibres over $\mathbf{C}^*$ using the map $(z_1, ..., z_{n+1}) \to z_0^2 + ... + z_{n+1}^2$. The fibre is homotopy equivalent to S^n. Hence the complement to the singular quadric can be identified with $S^1 \times S^n$. On the other hand the degeneration of the non-singular quadric into the singular one results in the collapse of the vanishing cycle S^n which is the boundary of a relative vanishing cycle. This relative vanishing cycle can be given explicitly as the set Δ of $(z_1, ..., z_{n+1}) \in \mathbf{R}^{n+1} \subset \mathbf{C}^{n+1}, |z_1|^2 + ... |z_{n+1}|^2 \le 1 - \epsilon)$. The complement to $z_0^2 + ... z_{n+1}^2 = 1$ hence can be obtained from the complement to $z_0^2 + ... z_{n+1}^2 = 0$ by attaching a $(n+1)$-cell. Therefore the complement to the non-singular quadric can be identified with $S^1 \times S^n \cup_{* \times S^n} e_{n+1} = S^1 \vee S^{n+1}$.

Definition 1.4. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ and H be a hyperplane. A point of V will be called a singular point at infinity if it is a singular point of $V \cap H$. The subvariety of singular points of V will be denoted $Sing_{\infty}(V)$.

Lemma 1.5. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ having the dimension of $Sing(V) \cup Sing_{\infty}(V)$ (resp. $Sing(V)$) equal to k. If $\mathbf{C}^{n-k+1}$ is a generic linear subspace of $\mathbf{C}^{n+1}$ of codimension k then $\mathbf{C}^{n-k+1} \cap V$ has isolated singularities and $\pi_{n-k}(\mathbf{C}^{n+1} - V) = \pi_{n-k}(\mathbf{C}^{n-k+1} - V \cap \mathbf{C}^{n-k+1})$ (resp. $\pi_{n-k}(\mathbf{CP}^{n+1} - (V \cup H)) = \pi_{n-k}(\mathbf{CP}^{n-k+1} - (V \cup H) \cap \mathbf{CP}^{n-k+1})$). Moreover $\pi_1(\mathbf{CP}^{n+1} - V \cup H) = \mathbf{Z}$ (resp. $\pi_1(\mathbf{CP}^{n+1} - V) = \mathbf{Z}/d\mathbf{Z}$) and $\pi_i(\mathbf{CP}^{n+1} - V \cup H) = \pi_i(\mathbf{CP}^{n+1} - V) = 0$) for $2 \leq i < n-k$.

Proof. The first part is a consequence of the Lefschetz theorem (cf. [H]). If L is a generic subspace of codimension $k+1$ in $\mathbf{CP}^{n+1}$ then $V \cap L$ is a non-singular hypersurface in L which is transversal to $L \cap H$ and the claim follows from the lemma 1.1.

Lemma 1.6. Let V be a hypersurface with isolated singularities including singularities at infinity. Then $H_i(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z})$ =0 for $2 \leq i \leq n-1$. If V is non-singular then the vanishing also takes place for $i = n$. If $n \geq 2$ then for $i = n$ this group is isomorphic to $H^{n+1}(V, H \cap V, \mathbf{Z})$. Moreover $H_1(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ unless $dim V = 1$, in which case this group is the free abelain group of the rank equal to the number of irreducible components of V.

Proof. The calculation of the group in the lemma for $i \leq n-1$ using the Lefschetz theorem can be reduced to the case $i = n$ (as was done above for homotopy groups). For $i = n$, as follows from the exact sequence of the pair $(\mathbf{CP}^{n+1} - H, \mathbf{CP}^{n+1} - (V \cup H))$ the group in the lemma is isomorphic to $H_{n+1}(\mathbf{CP}^{n+1} - H, \mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z})$. If $T(V)$ and $T(H)$ are the regular neighbourhoods of V and H respectively in $\mathbf{CP}^{n+1}$ and ∂_{∞} is intersection of the boundary of $T(H)$ with $T(V) - T(V) \cap T(H)$ then the last homology group can be replaced using excision and the Lefschetz duality by $H^{n+1}(T(V), \partial_{\infty}, \mathbf{Z})$. If

V is non-singular then the latter group can be identified with $H_{n-1}(V - V \cap H, \mathbf{Z}) = 0$, which proves the lemma for $2 \leq i \leq n-1$. In the general case the last cohomology group by excision is isomorphic to $H^{n+1}(V, H \cap V, \mathbf{Z})$ which proves the lemma. The remaining case of curves is well known (cf. [L]).

Lemma 1.7. If V and $V \cap H$ are $\mathbf{Q}$-manifold then $H^{n+1}(V, H \cap V, \mathbf{Q}) = 0$.

Proof. This is equivalent to injectivity of $H^i(V, \mathbf{Q}) \to H^i(H, \mathbf{Q})$ for $i = n+2$ and surjectivity for $i = n+1$. This follows from the Poincare duality with $\mathbf{Q}$-coefficients and the Lefschetz theorem (the dual homology groups are isomorphic to $\mathbf{Q}$ or to 0 depending on parity of n).

Remark 1.8. An easily verifyable condition for a hypersurface V to be a $\mathbf{Q}$-manifold is the following: if V has only isolated singularities and for each of the singularities the characteristic polynomial of the monodromy operator does not vanish at 1 then V is a $\mathbf{Q}$-manifold. This is an immediate consequence of the fact that the condition on the monodromy is equivalent to the condition that the link of each singualarity is a $\mathbf{Q}$ -sphere and of the Zeeman spectral sequence (cf. [McC]).

Let V be a hypersurface in $\mathbf{C}^{n+1}$ having only isolated singularities including infinity. According to lemma 1.5, one can identify $\pi_n(\mathbf{C}^{n+1} - V)$ with $H_n(\widetilde{\mathbf{C}^{n+1} - V}, \mathbf{Z})$ where $\widetilde{\mathbf{C}^{n+1} - V}$ is the universal cover of the space in question. The group $\mathbf{Z}$ of deck transformations acts on $H_n(\widetilde{\mathbf{C}^{n+1} - V}, \mathbf{Z})$. This action on $\pi_n(\mathbf{C}^{n+1} - V)$ can be described as $\beta \to [\alpha, \beta] - \beta$ where $\beta \in \pi_n(\mathbf{C}^{n+1} - V)$, $\alpha \in \pi_1(\mathbf{C}^{n+1} - V)$ and $[,]$ is the Whitehead product (alternatively this action is the one given by the change of the base point). The structure of $\pi_n(\mathbf{C}^{n+1} - V)$ as the module over the group ring of the fundamental group, i.e. over $\mathbf{Z}[t, t^{-1}]$, becomes particularly simple after tensoring with $\mathbf{Q}$:

$$\pi_n(\mathbf{C}^{n+1} - V) \otimes \mathbf{Q} = \oplus_i \mathbf{Q}[t, t^{-1}]/(\lambda_i) \tag{1.1}$$

as modules over $\mathbf{Q}[t,t^{-1}]$ for some polynomials λ_i defined up to a unit of $\mathbf{Q}[t,t^{-1}]$.

Definition 1.9. The product $\prod_i \lambda_i$ is called the order of $\pi_n(\mathbf{C}^{n+1}-V)\otimes\mathbf{Q}$ (as a module over $\mathbf{Q}[t,t^{-1}]$).

Remark 1.10. In the low dimensional cases, if one works with the homology of infinite cyclic covers one obtains results similar to what follows. The order of the corresponding $\mathbf{Q}[t,t^{-1}]$ module in the case $n=1$ is the Alexander polynomial of the curves studied in [L2]. Note that if $n=0$ then the one-dimensional homology over $\mathbf{Z}$ of the infinite cyclic cover of $\mathbf{C}-V$ (i.e. of the complement to, say, d points) is a free module over $\mathbf{Z}[t,t^{-1}]$ of rank $d-1$.

Lemma 1.11. Let $f(z_1,...,z_n)$ be a weighted homogeneous polynomial having an isolated singularity at the origin and V is given by $f=0$. Then $\pi_n(\mathbf{C}^{n+1}-V)=H_n(M_f,\mathbf{Z})$ where M_f is the Milnor fibre of the singularity of f. This isomorphism is an isomorphism of $\mathbf{Z}[t,t^{-1}]$-modules where the structure of such module on $H_n(M_f,\mathbf{Z})$ is given t acting as the monodromy operator.

Proof. $\mathbf{C}^{n+1}-V$ can be retracted on the complement of the link of the singularity of f. Hence this follows from the exact sequence of fibration and $n-1$-connectedness of the Milnor fibre.

Lemma 1.12. Let $P_V(t)$ be the order of $\pi_n(\mathbf{CP}^{n+1}-(V\cup H))\otimes\mathbf{Q}$ as $\mathbf{Q}[t,t^{-1}]$-module. If $H^{n+1}(V,H\cap V,\mathbf{Q})=0$ then $P_V(1)\neq 0$. In particular this homotopy group is a torsion module.

Proof. Let us consider the exact sequence corresponding to the following exact sequence of the chain complexes of the universal cyclic cover $(\mathbf{CP}^{n+1}\widetilde{-(V\cup H)})_{\infty}$:

$$0\to C_*((\mathbf{CP}^{n+1}\widetilde{-(V\cup H)})_{\infty})\to C_*((\mathbf{CP}^{n+1}\widetilde{-(V\cup H)})_{\infty})\to C_*(\mathbf{CP}^{n+1}-(V\cup H))\to 0 \tag{1.2}$$

where the left homomorphism is the map of free $\mathbf{Q}[t,t^{-1}]$- modules induced by the multiplication by $t-1$. We obtain:

$$\to H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty, \mathbf{Q}) \to H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty, \mathbf{Q}) \to H_n(\mathbf{CP}^{n+1} - (V\cup H), \mathbf{Q}) \to \tag{1.3}$$

(the left homomorphism is the multiplication by $t-1$.) The right group in (1.3) is trivial by the assumption and lemma 1.6. Hence the multiplication by $t-1$ in $\pi_n(\mathbf{CP}^{n+1} - (V\cup H)) \otimes \mathbf{Q} = H_n(\widetilde{\mathbf{CP}^{n+1} - (V\cup H)}\infty, \mathbf{Q})$ is surjective. Therefore, its cyclic decomposition does not have either free summands or summands of the form $\mathbf{Q}[t,t^{-1}]/(t-1)^\kappa\mathbf{Q}[t,t^{-1}]$ $\kappa \in \mathbf{N}$.

Lemma 1.13. Let H be a generic hyperplane and V a hypersurface of degree d with isolated singularities in $\mathbf{CP}^{n+1}$. Let $d\mathbf{Z}$ be the subgroup of $\mathbf{Z}$ of index d. Then $\pi_n(\mathbf{CP}^{n+1} - V)$ is isomorphic to the covariantes of $\pi_n(\mathbf{CP}^{n+1} - (V \cup H))^{d\mathbf{Z}}$ (i.e. the quotient by the submodule of images by the action of augmentation ideal of the subgroup: $\pi_n/(t^d-1)\pi_n$) with the standard action of $\mathbf{Z}/d\mathbf{Z}$.

Proof. First let us show that the module of covariants in the lemma is isomorphic to

$$H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_d, \mathbf{Z}) \tag{1.4}$$

where $(\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_d$ is the d-fold cyclic cover of the corresponding space. Indeed the sequence of the chain complexes similar to (1,2):

$$0 \to C_*((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty) \to C_*((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty) \to C_*((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_d) \to \tag{1.5}$$

where the left homomorphism is the multiplication by t^d-1, gives rise to the homology sequence:

$$\to H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty, \mathbf{Z}) \to H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_\infty, \mathbf{Z}) \to H_n((\widetilde{\mathbf{CP}^{n+1} - (V\cup H)})_d, \mathbf{Z}) \to \tag{1.6}$$

in which the right homomorphism is surjective because of the vanishing of $\pi_{n-1}(\mathbf{CP}^{n+1} - (V \cup H))$, which proves our claim.

To conclude the proof of the lemma we need to show that

$$H_n((\mathbf{CP}^{n+1} \widetilde{- (V \cup H)})_d, \mathbf{Z}) = H_n((\mathbf{CP}^{\widetilde{n+1}} - V)_d, \mathbf{Z}) \tag{1.7}$$

where $(\mathbf{CP}^{\widetilde{n+1}} - V)_d$ is the universal cyclic cover of $\mathbf{CP}^{n+1} - V$. This will follow from the vanishing of the relative group $H_i(\widetilde{(}\mathbf{CP}^{n+1} - V)_d, (\widetilde{(}\mathbf{CP}^{n+1} - (V \cup H))_d, \mathbf{Z})$ for $i = n, n+1$. Let W_d be the cyclic d-fold branched over V covering of $\mathbf{CP}^{n+1}$ and let $Z \subset W_d$ be the ramification locus (isomorphic to V). Let H_d be the submanifold of W_d which maps onto $H \cap V$. Let us consider the regular neighbourhoods $T(H_d)$ and $T(Z)$ in W_d of H_d and Z respectively. The boundary of $T(H_d) - T(H_d) \cap T(Z)$ contains the part ∂_1 which is the part of the boundary of $T(H_d)$. The complementary to ∂_1 part of the boundary of $T(H_d) - T(H_d) \cap T(Z)$ we denote ∂_2. By excision:

$$H_i((\mathbf{CP}^{\widetilde{n+1}} - V)_d, (\mathbf{CP}^{n+1} \widetilde{- (V \cup H})_d, \mathbf{Z}) = H_i(T(H_d) - T(H_d) \cap T(Z), \partial_1, \mathbf{Z}) \tag{1.8}$$

Moreover $H_i(T(H_d) - T(H_d) \cap T(Z), \partial_1, \mathbf{Z}) = H^{2n+2-i}(T(H_d) - T(H_d) \cap T(Z), \partial_2, \mathbf{Z}) = H^{2n+2-i}(H_d, H_d \cap Z, \mathbf{Z})$ by duality, excision and retraction. The assumption that H is generic implies that H_d,which is a cyclic branched cover of H of degree d with branching locus $H \cap Z$, is non-singular and hence $H^{2n+2-i}(H_d, H_d \cap Z, \mathbf{Z}) = H_{i-2}(H_d - H_d \cap Z, \mathbf{Z})$. The latter group which is the homology group of affine hypersurface transversal to the hyperplane at infinity is trivial except for $i = 2$ and $i = n + 2$. This implies the lemma.

2. Calculation of the homotopy groups using generic pencils.

In this section we shall describe a method for calculation of $\pi_n(\mathbf{C}^{n+1} - V)$ using the monodromy action on the homotopy groups of the complement in a generic element of a

linear pencil of hyperplanes to the intersection of this element with V obtained by moving this element around a loop in the parameter space of the pencil. Monodromy action is the composition of "geometric monodromy" with values in the fundamental group of certain embeddings and a linear representation of this group over $\mathbf{Z}[t, t^{-1}]$. In the case $n = 1$ this construction with monodromy taking values in the fundamental group of the space of embeddings (which in this case is the Artin's braid group), reduces to the van Kampen theorem (cf. [vK]). The composition of this monodromy with the Burau representation leads to calculation of the Alexander module of the curve. We will start with specifying the loops such that by moving along these loops we will get the information needed.

Definition 2.1. Let $t_1, ..., t_N$ be a finite set of points in $\mathbf{C}$. A system of generators of $\gamma_i \in \pi_1(\mathbf{C} - \bigcup_i t_i, t_0)$ is called *good* if each of the loops $\gamma_i : S^1 \to \mathbf{C} - \bigcup_i t_i$ extends to a map of the disk $D^2 \to \mathbf{C}$ with non-intersecting images for distinct i's.

A standard method for constructing a good system of generators is to select a system of small disks Δ_i about each of t_i $i = 1, ..., N$, to choose a system of N non-intersecting paths δ_i connecting the base point t_0 with a point of $\partial\Delta_i$ and to take $\gamma_i = \delta^{-1} \circ \partial\Delta_i \circ \delta_i$ (with, say, the counterclockwise orientation of $\partial\Delta_i$).

Let V be a non-singular hypersurface in $\mathbf{C}^n$ which is transversal to the hyperplane at infinity. Let us consider a sphere in S^{2n-1} in $\mathbf{C}^n$ of a sufficiently large radius. Let $\partial_\infty V = V \cap S^{2n-1}$. Let us consider the space of $Emb(V, \mathbf{C}^n)$ of submanifolds of $\mathbf{C}^n$ which are diffeomorphic to V and which are isotopic to the chosen embedding of V, such that for any $V' \in Emb(V, \mathbf{C}^n)$ one has $V'(V) \cup S^{2n-1} = \partial_\infty V$. We assume the compact open topology on this space of submanifolds. Let us describe a certain linear representation of $\pi_1(Emb(V, \mathbf{C}^n))$ (over $\mathbf{Z}[t, t^{-1}]$) which after a choice of a basic gives homomorphism into $GL_r(\mathbf{Z}[t, t^{-1}])$ where r is the rank of $\tilde{H}_n(\mathbf{C}^n - V, \mathbf{Z})$ (the reduced homology of the complement).

Let $Diff(\mathbf{C}^n, S^{2n-1})$ be the group of diffeomorphisms of $\mathbf{C}^n$ which act as the identity

outside of S^{2n-1}. This group can be identified with $Diff(S^{2n}, D_{2n})$ of the diffeomorphisms of the sphere fixing a disk (cf. [ABK]). Let $Diff(\mathbf{C}^n, V)$ be the subgroup of $Diff(\mathbf{C}^n, S^{2n})$ of the diffeomorphisms which take V into itself. The group $Diff(\mathbf{C}^n, S^{2n-1})$ acts transitively on $Emb(V, \mathbf{CP}^n)$ with the stabilizer $Diff(\mathbf{C}^n, V)$ which implies the following exact sequence:

$$\pi_1(Diff(S^{2n+2}, D_{2n+2})) \to \pi_1(Emb(\mathbf{C}^{n+1}, V)) \to \pi_0(Diff(\mathbf{C}^{n+1}, V)) \to \pi_0(Diff(S^{2n+2}, D^{2n+2})) \to \tag{2.1}$$

Any element in $Diff(\mathbf{C}^n, V)$ induces the self map of $\mathbf{C}^n - V$ and the self map of the universal (cyclic in the case $n = 1$) cover of this space. Hence it induces an automorphism of $H_n(\widetilde{\mathbf{C}^n - V}, \mathbf{Z}) = \pi_n(\mathbf{C}^n - V)$, $n > 1$. . The composition of the boundary homomorphism in (2.1) with the map of $\pi_0(Diff(\mathbf{C}^n, V))$ just described results in:

$$\lambda : \pi_1(Emb(\mathbf{C}^n, V)) \to Aut(\pi_n(\mathbf{C}^n - V)) \tag{2.2}$$

In the case $n = 1$, V is just a collection of points in $\mathbf{C}$, $\pi_1(Emb(\mathbf{C}, V)) = \pi_0(Diff(\mathbf{C}, V))$ is Artin's braid group, and this construction gives the homomorphism of the braid group into $Aut(H_1(\widetilde{\mathbf{C} - V}, \mathbf{Z})$ which, after a choice of the basis in $H_1(\widetilde{\mathbf{C} - V})$ corresponding to the choice of the generators of the braid group, gives the reduced Burau representation. This construction coincides with the one described in [A].

Now we can define the relevant monodromy operator corresponding to a loop in the parameter space of a linear pencil of hyperplane sections. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which has only isolated singularities and H be the hyperplane at infinity (which we shall assume transversal to V). Let L_t, $t \in \mathbf{C}$, be a pencil of hyperplanes the projective closure of which has the base locus $B \subset H$ such that B is transversal to V. Let $t_1, ..., t_N$ denote those t for which $V \cap L_t$ has a singularity. We assume that for any i the singularity of $V \cap L_{t_i}$ is outside of H. Over $\mathbf{C} - \bigcup_i t_i$ the pencil L_t defines a locally trivial fibration τ of $\mathbf{C}^{n+1} - V$ with a non-singular hypersurface in $\mathbf{C}^n$ as a fibre transversal to the hyperplane at infinity. The restriction of this fibration on the complement to a sufficiently large ball

is trivial, as follows from the assumptions on the singularities at infinity. Let $\gamma : [0,1] \rightarrow \mathbf{C} - \bigcup_i t_i \ (i = 1, ..., N)$ be a loop with the base point t_0. A choice of a trivialization of the pull back of the fibration τ on $[0,1]$ using γ, defines a loop e_γ in $Emb(L_{t_0}, V \cap L_{t_0})$. Different trivializations define homotopic loops in this space.

Definition 2.2. The monodromy operator corresponding γ is the element in $Aut(\pi_n(L_{t_0} - L_{t_0} \cap V))$ corresponding in (2.2) to e_γ.

Next we will need to associate the homomorphism with a singular fibre L_{t_i} and a loop γ with the base point t_0 in the parameter space of the pencil which there bounds a disk Δ_{t_i} not containing other singular points of the pencil :

$$\pi_{n-1}(L_{t_i} - L_{t_i} \cap V) \rightarrow \pi_n(L_{t_0} - L_{t_0} \cap V)/Im(\Gamma - I) \tag{2.3}$$

where Γ is the monodromy operator corresponding to γ.

First let us note that the module on the right in (2.3) is isomorphic to the homology $H_n(\widetilde{\tau^{-1}(\partial\Delta_{t_i})}, \mathbf{Z})$ of the infinite cyclic cover of the restriction of the fibration τ on the boundary of Δ_{t_i}. This follows immediately from the Wang sequence of a fibration over a circle and the vanishing of the homotopy of $L_{t_0} - L_{t_0} \cap V$ in dimensions below n. Let B_i be a polydisk in $\mathbf{C}^{n+1}$ such that $B_i = \Delta_i \times B$ for a certain polydisk B in L_{t_0}. Then $\widetilde{\tau^{-1}(\Delta_i)} - B_i$ is a trivial fibration over Δ_i with the infinite cyclic cover $\widetilde{L_{t_i} - L_{t_i}} \cap V$ as a fibre. In particular, one obtains the map:

$$\pi_{n-1}(L_{t_0} - L_{t_0} \cap V) = H_{n-1}(\widetilde{L_{t_0} - L_{t_0}} \cap V, \mathbf{Z}) \rightarrow H_n(\widetilde{\tau^{-1}(\Delta_i)} - B_i, \mathbf{Z}) =$$

$$H_{n-1}(\widetilde{L_{t_0} - L_{t_0}} \cap V, \mathbf{Z}) \oplus H_n(\widetilde{L_{t_0} - L_{t_0}} \cap V, \mathbf{Z}) \tag{2.4}$$

Definition 2.3. The degeneration operator is the map (2.3) given by composition of the map (2.4) with the map $H_n(\widetilde{\tau^{-1}(\Delta_i)} - B_i, \mathbf{Z}) \rightarrow H_n(\widetilde{\tau^{-1}(\Delta_i)}, \mathbf{Z}) = \pi_n(L_{t_0} - L_{t_0} \cap V)$ induced by inclusion.

Theorem 2.4. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ having only isolated singularities and transversal to the hyperplane H at infinity. Consider a pencil of hyperplanes in $\mathbf{CP}^{n+1}$

the base locus of which belongs to H and is transversal in H to $V \cap H$. Let $\mathbf{C}^n_t$ $(t \in \Omega)$ be the pencil of hyperplanes in $\mathbf{C}^{n+1} = \mathbf{CP}^{n+1} - H$ defined by L_t (where $\Omega = \mathbf{C}$ is the set parametrizing all elements of the pencil L_t excluding H). Denote by $t_1, ..., t_N$ the collection of those t for which $V \cap L_t$ has a singularity. We shall assume that the pencil was chosen so that $L_t \cap H$ has at most one singular point outside of H. Let t_0 be different from either of t_i $(i = 1, ..., N)$. Let γ_i $(i = 1, ..., N)$ be a good collection, in the sense described above (Def.(2.1)), of paths in Ω based in t_0 and forming a basis of $\pi_1(\Omega - \bigcup_i t_i, t_0)$ and let $\Gamma_i \in Aut(\pi_n(\mathbf{C}^n_t - V \cap \mathbf{C}^n_t))$ be the monodromy automorphism corresponding to γ_i. Let $\sigma_i : \pi_{n-1}(\mathbf{C}^n_{t_i} - V \cap \mathbf{C}^n_{t_i}) \to \pi_n(\mathbf{C}^n_{t_0} - V \cap \mathbf{C}^n_{t_0})^{\Gamma_i}$ be the degeneration operator of the homotopy group of a special element of the pencil into the correponding quotient of covariants constructed above. Then

$$\pi_n(\mathbf{C}^{n+1} - V \cap \mathbf{C}^{n+1}) = \pi_n(\mathbf{C}^n - V \cap \mathbf{C}^n)/(Im(\Gamma_1 - I), Im\sigma_1, ..., Im(\Gamma_N - I), \sigma_N) \quad (2.5)$$

Proof. Let $P : \mathbf{C}^{n+1} \to \Omega$ be the projection defined by the pencil $\mathbf{C}_t$. Let $T(\mathbf{C}^n_{t_i})$ be the intersection of the tubular neighbourhood of L_i in $\mathbf{CP}^{n+1}$ with the finite part $\mathbf{C}^{n+1}$ which can be taken as $P^{-1}(\Delta_i)$ where $\Delta_i \subset \Omega$ is a small disk about t_i $(i = 1, ..., N)$. Each loop γ_i is isotopic to the loop having the standard form $\delta_i^{-1} \circ \partial\Delta_i \circ \delta_i$ (δ_i, as above, is a system of paths in Ω connecting t_0 to $\partial\Delta_i$ and non-intersecting outside of t_0). We shall assume from now on that γ_i's have such form. The restriction P_V of P on $\mathbf{C}^{n+1} - V \cap \mathbf{C}^{n+1}$ defines over $\Omega - \bigcup_i(\gamma_i \cup \Delta_i)$ a locally trivial fibration and therefore $\mathbf{C}^{n+1} - V \cap \mathbf{C}^{n+1}$ is homotopy equivalent to $P_V^{-1}(\bigcup_i(\gamma_i \cup \Delta_i))$. The latter space is homotopy equivalent to

$$\bigcup_{T(\mathbf{C}^n_{t_0}) - V \cap T(\mathbf{C}^n_{t_0})} T(\mathbf{C}^n_{t_i}) - V \cap T(\mathbf{C}^n_{t_i})(i = 1, ..., N) \quad (2.6)$$

with the embedding of the common part of the spaces in the union in each of them depending on the trivialization of $P_V^{-1}(\delta_i)$ over δ_i. We are going to calculate the homology of the infinite cyclic cover of (2.6) by repeated use of the Mayer-Vietoris sequences. First

we claim that if $t_{0,i}$ denotes the end point of the path δ_i and Γ'_i is the authomorphism of $\pi_n(\mathbf{C}^n_{t_{0,i}} - \mathbf{C}^n_{t_{0,i}} \cap V)$ induced by the monodromy corresponding to the loop $\partial\Delta_i$ then:

$$H_n((T(\mathbf{C}^n_{t_i} - \widetilde{V \cap T}(\mathbf{C}^n_{t_i}), \mathbf{Z}) = \pi_n(\mathbf{C}^n_{t_{0,i}} - V \cap \mathbf{C}^n_{t_{0,i}})/(Im(\Gamma'_i - I), Im\sigma'_i) \tag{2.7}$$

for any i $(i = 1, ..., N)$ where $\widetilde{X}$ denotes the universal cyclic cover of a space X . To verify (2.7) let us consider a small polydisk $B_i \subset T(\mathbf{C}^n_{t_i}$ for which the projection P induces a split $B_i = \Delta'_i \times D^n_i$ as a product of a 2-disk $\Delta'_i \subset \Delta_i$, $t_i \in \Delta'_i$ and n-disk D^n_i in $\mathbf{C}^n_{t_i}$ such that D^n_i contains the singular point of $V \cap \mathbf{C}^n_{t_i}$. One has a natural retraction $B_i - B_i \cap V$ onto $\partial B_i - \partial B_i \cap V$ which shows that $T(\mathbf{C}^n_{t_i}) - T(\mathbf{C}^n_{t_i}) \cap V$ is homotopy equivalent to $T(\mathbf{C}^n_{t_i}) - T(\mathbf{C}^n_{t_i}) \cap V - B$. Let us decompose the latter as :

$$P_V^{-1}(\Delta_i - \Delta'_i) \cup (P_V^{-1}(\Delta'_i) - B_i) \tag{2.8}$$

The first component in this union, which we shall call Θ_1, is a locally trivial fibration over homotopy circle $\Delta_i - \Delta'_i$ with fibre $\mathbf{C}^n_{t_0} - V \cap \mathbf{C}^n_{t_0}$. The second, which we denote Θ_2, is fibred over 2-disk Δ'_i with the fibre $\mathbf{C}^n_{t_i} - \mathbf{C}^n_{t_i} \cap V$ and hence is homotopy equivalent to this fibre. The intersection Θ_0 of two pieces in (2.8) which is the preimage of the boundary circle $\partial\Delta'_i$ forms a part of a fibration over a disk and hence is homotopy equivalent to $\mathbf{C}^n_{t_i} - \mathbf{C}^n_{t_i} \cap V$. This decomposition defines the decomposition of the infinite cyclic covers:

$$T(\mathbf{C}^n_{t_i}) - \widetilde{T(\mathbf{C}^n_{t_i}} \cap V) = \tilde{\Theta}_1 \bigcup_{\tilde{\Theta}_0} \tilde{\Theta}_2 \tag{2.9}$$

The split of Ω_0 as a direct product $\mathbf{C}^n_{t_i} - \mathbf{C}^n_{t_i} \cap V \times S^1$ implies the splitting of the infinite cyclic cover of $\tilde{\Omega}_0$ as $(\mathbf{C}^n_{t_i} - \widetilde{\mathbf{C}^n_{t_i}} \cap V) \times S^1$. Therefore

$$H_j(\tilde{\Omega}_0, \mathbf{Z})) = H_{j-1}(\mathbf{C}^n_{t_i} - \widetilde{\mathbf{C}^n_{t_i}} \cap V, \mathbf{Z}) \oplus H_j(\mathbf{C}^n_{t_i} - \widetilde{\mathbf{C}^n_{t_i}} \cap V, \mathbf{Z})(j \in \mathbf{Z}) \tag{2.10}$$

Next let us consider the Mayer-Vietoris homology sequence corresponding to decomposition (2.6):

$$\rightarrow H_n(\tilde{\Omega}_0, \mathbf{Z}) \rightarrow H_n(\tilde{\Omega}_1, \mathbf{Z}) \oplus H_n(\tilde{\Omega}_2, \mathbf{Z}) \rightarrow H_n(T(\mathbf{C}^n_{t_i}) - \widetilde{T(\mathbf{C}^n_{t_i}}) \cap V, \mathbf{Z}) \rightarrow$$

$$\to H_{n-1}(\tilde{\Omega}_0) \to H_{n-1}(\tilde{\Omega}_1) \oplus H_{n-1}(\tilde{\Omega}_2) \tag{2.11}$$

Th group $H_n(\tilde{\Omega}_1, \mathbf{Z})$ can be identified as above with $\pi_n(\mathbf{C}^n_{t_{0,i}} - \mathbf{C}^n_{t_{0,i}} \cap V)/(\Gamma'_i - I)$ and the left homomorphism in (2.11) takes the second summand in (2.10) in the case $j = n$ isomorphically to $H_n(\tilde{\Omega}_2, \mathbf{Z})$. Hence the *Coker* of the left homomorphism in (2.11) coincides with $\pi_n(\mathbf{C}^n_{t_{0,i}} - \mathbf{C}^n_{t_{0,i}} \cap V)/(\Gamma'_i - I, Im\sigma'_i)$. Moreover the same argument shows that the homomorphism of $H_{n-1}(\tilde{\Omega}_0, \mathbf{Z})$ in (2.11) is an injection because $H_{n-2}(\widetilde{\mathbf{C}^n_{t_{0,i}} - \mathbf{C}^n_{t_{0,i}}}, \mathbf{Z})$ is isomorphic to π_{n-2} of the same space and therefore is trivial (cf. Lemma (1.5)) which implies (2.7).

Next we shall calculate the homology of the infinite cyclic cover in (2.6) inductively using the Mayer-Vietoris sequence corresponding to this decomposition. Because $\pi_i(\mathbf{C}^n_{t_0} - \mathbf{C}^n_{t_0} \cap V) = 0$ for $2 \leq i \leq n-1$ (cf. lemma (1.5)), the terms in the Mayer-Vietoris sequence below dimension n vanish. The cokernel of the map:

$$H_n(T(\mathbf{C}^n_{t_0}) - \widetilde{T(\mathbf{C}^n_{t_0}) \cap V}, \mathbf{Z}) \to$$

$$\to H_n(\mathbf{C}^n_{t_{0,i}} - \widetilde{\mathbf{C}^n_{t_{0,i}} \cap V}, \mathbf{Z})/(Im(\Gamma'_i - I), Im\sigma'_i) \oplus H_n(\mathbf{C}^n_{t_{0,j}} - \widetilde{\mathbf{C}^n_{t_{0,j}} \cap V}, \mathbf{Z})/(Im(\Gamma'_j - I), Im\sigma'_j) \tag{2.12}$$

by the linear algebra is isomorphic to $H_n(\mathbf{C}^n_{t_0} - \widetilde{\mathbf{C}^n_{t_0} \cap V}, \mathbf{Z})/Im(\Gamma_i - I), Im\sigma_i, Im(\Gamma_j - I), Im\sigma_j)$. Now the theorem follows.

3. A vanishing theorem.

In this section we give a necessary condition for the vanishing of $\pi_n(\mathbf{C}^{n+1} - V)$. This is useful in applications of theorem 2.4 because it allows one to dispose of degeneration operator in some cases. We give a numerical consequence in the case of curves. The key part is the following counterpart of the commutativity of the fundamental group in the case of curves.

Theorem 3.1. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which has only isolated singularities including singularities at infinity, and let H be the hyperplane at infinity. Suppose that there is a non singular variety W and a map $\phi : W \to \mathbf{CP}^{n+1}$ such that the union of the proper preimage V' of V,the proper preimage H' of H, and the exceptional locus E of ϕ, form a divisor with normal crossings on W. Assume that V' is an *ample* divisor on W. Then the action of $\pi_1(\mathbf{CP}^{n+1} - (V \cup H) = \mathbf{Z}$ on $\pi_n(\mathbf{CP}^{n+1} - (V \cup H))$ is trivial.

Proof. Let $U_{V'}$ be a tubular neighbourhood of V' in W. First note that the map $\pi_i((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V')) \to \pi_i(W - (E \cup V' \cup H'))$ induced by inclusion is an isomorphism for $i \leq n-1$ and is surjective for $i = n$. This follows immediately from the assumption of ampleness of V' and the Lefschetz theorem for open varieties (cf. [H]. In the setting of this work one applies the theorem 2 from this paper to W embedded into $\mathbf{CP}^N$ using a multiple of the line bundle corresponding to V' and taking V' as the hyperplane section at infinity). Because $W - (V' \cup H' \cup E) = \mathbf{CP}^{n+1} - (V \cup H)$ we obtain $\psi_i : \pi_i((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V')) \to \pi_i(\mathbf{CP}^{n+1} - (V \cup H)$ which is an isomorphism for $i \leq n-1$ and surjective for $i = n$.

Let α be the boundary of the normal to V' 2-disk in $U_{V'}$ at a point of V' outside of $E \cup H'$. The action of α considered as an element of $\pi_1((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V'))$ on $\pi_n((U_{V'} - V') - (E \cup H') \cap (U_{V'} - V'))$ is trivial. Indeed $(U_{V'} - V') - (U_{V'} - V') \cap (E \cup H')$ is a (trivial) circle bundle over $V' - (V' \cap (E \cup H')$ because $V' \cup H' \cup E$ is assumed to be a divisor with normal crossings. Moreover the projection map λ induces the isomorphism $\lambda_* : \pi_i((U_{V'} - V') - (U_{V'} - V') \cap (E \cup H')) \to \pi_i(V' - V' \cap H' \cap E)$ for $i \geq 2$. If $\gamma \in \pi_i((U_(V') - V') - (U_(V') - V') \cap (E \cup H'))$ then $\lambda_*(\alpha \cdot \gamma) = \lambda_*(\alpha) \cdot \lambda_*(\gamma) = \lambda_*(\gamma)$ i.e. $\alpha \cdot \gamma = \gamma$ and our claim follows. This also concludes the proof of the theorem because ψ_n is surjective and because α is taken by ψ_1 into the generator of $\pi_1(\mathbf{CP}^{n+1} - (V \cup H))$.

Theorem 3.2. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which satisfies all conditions of the theorem 3.1. Let us assume also that $H^{n+1}(V, V \cap H, \mathbf{Z}) = 0$. Then $\pi_n(\mathbf{CP}^{n+1} - (V \cup H))$

vanishes.

Proof. Let us consider the Leray spectral sequence

$$E^2_{p,q} = H_p(\mathbf{Z}, H_q((\widetilde{\mathbf{CP}^{n+1} - (V \cup H)}, \mathbf{Z})) \Rightarrow H_{p+q}(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z}) \qquad (3.1)$$

asssociated with the classifying map of $\mathbf{CP}^{n+1} - (V \cup H)$ into $S^1 = K(\mathbf{Z}, 1)$ corresponding to the generator of $H^1(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z}) = \mathbf{Z})$ (the homotopy fibre of this map is the universal cyclic cover $(\widetilde{\mathbf{CP}^{n+1} - (V \cup H)})$. It implies that $H_0(\mathbf{Z}, H_n((\widetilde{\mathbf{CP}^{n+1}} - (V \cup H)), \mathbf{Z})$ which isomorphic to the covariants $H_n((\widetilde{\mathbf{CP}^{n+1} - (V \cup H)}), \mathbf{Z})^{\pi_1(\mathbf{CP}^{n+1} - (V \cup H))} = H_n(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z})$. The group in the left hand side is isomorphic to the covariants $\pi_n(\mathbf{CP}^{n+1} - (V \cup H))^{\pi_1(\mathbf{CP}^{n+1} - (V \cup H))}$. Hence the result follows from the above theorem, and the vanishing of $H_n(\mathbf{CP}^{n+1} - (V \cup H), \mathbf{Z})$ which is a consequence of the lemma 1.6 and the assumption we made on the cohomology of $(V, V \cap H)$.

Corollary 3.3. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ which satisfies the condition of the theorem 3.1. If V and $V \cap H$ are $\mathbf{Q}$-manifolds then $\pi_n(\mathbf{CP}^{n+1} - (V \cap H)) = 0$.

Proof. This follows from 3.2 and 1.7.

Remark 3.3 In the case $n = 1$ the argument given in the proof of the theorem 3.1 can be strengthened to show that $(V \cdot V$ implies that the fundamental group of the complement is abelian (cf. [Ab],[N]). Recall that, for example for a curve C of degree d which has δ nodes and κ cusps, this implies the commutativity of $\pi_1(\mathbf{CP}^{n+1} - C)$ provided $4\delta + 6\kappa < d^2$. For the application of the technique of section 2 the following result is useful.

Lemma 3.4 Let C be a curve in $\mathbf{CP}^2$ which has only one singular point which is unibranched and has one characteristic pair (m, k) $(k \leq m)$. If $d^2 > m \cdot k$ then $\pi_1(\mathbf{CP}^2 - C)$ is abelian. In particular the Alexander module of C is trivial.

Proof. The greatest common divisor of m and k which is equal to the number of branches of the singularity is equal to 1. The resolution of singularities of plane curves can be described in terms of the Euclidian algorithm for finding this greatest common divisor (cf.[BK]). Let $m = a_1 k + r_1, k = a_2 \cdot r_1 + r_2, ...r_{s-1} = a_{s+1} \cdot r_s + 1$ be the steps of the Euclidain algorithm. Then the sequence of blow ups which results in the embedded resolution produces the following. Each of the first a_1 blow ups gives an exceptional curve with multiplicitity k, the intersection index of which with the proper preimage of the curve in quesiton is equal to k. This results in dropping of the self-intersection index of the proper preimage by k^2. Subsequent blow ups have a similar effect with the blow ups corresponding to the last step in Euclidian algorithm resulting in a non-singular proper preimage with the tangency order with the exceptional curve equal to r_s. Additional r_s blow ups result in a proper preimage, the union of which with the exceptional locus, is a divisor with normal crossings. The self intersection of the proper preimage is equal to $d^2 - a_1 \cdot k^2 - a_1 \cdot r_1^2 - ... - a_{s+1} \cdot r_s^2 - r_s = d^2 - (m - r_1) \cdot k - ... - (r_{s+1} - 1) \cdot r_s - r_s = d^2 - m \cdot k > 0$. Hence our claim follows from the Nori's theorem [N].

4. The divisibility theorems.

In this section we prove two theorems relating the order of the homotopy group of the complement to the hypersurface V to the type of the singlarities V including the singularities of V at infinity. We will discuss the relation of these results to the divisibility theorem for Alexander polynomials in [L]. We asssume as above that V has only isolated singularities.

First recall that if $c \in V$ is a singular point of V then one can associate with it the characteristic polynomial P_c of the monodromy operator in the Milnor fibration of the singularity c. By a certain abuse of language we will call this polynomial the polynomial of the singularity c. The cyclic cover U_c of $B_c - B_c \cap V$ corresponding to the kernel of

the homomorphism $lk_c : \pi_1(B_c - B_c \cap V) \to H_1(B_c - B_c \cap V, \mathbf{Z}) = \mathbf{Z}$ (it is given by the evaluation of the linking number with $V \cap B_c$) is homotopy equivalent to the Milnor fibre and the characteristic polynomial of the automorphism of $H_n(U_c, \mathbf{Q})$ induced by the deck transformation coincides with the polynomial of the singularity c as a consequence of existence of fibration of $B_c - B_c \cap V$. On the other hand if $c \in Sing_\infty$ and B_c is a small ball in a certain Riemannian metric in $\mathbf{CP}^{n+1}$ about c, then $H_1(B_c - (V \cup H) \cap B_c, \mathbf{Z}) = H_2(B_c, B_c - (V \cup H), \mathbf{Z}) = H^{2n}(T(V \cup H), S_c \cap T(V \cup H)$ where S_c is the boundary of B_c and $T()$ denotes the regular neighbourhood. The latter is isomorphic to $H^{2n}(V \cap B_c, \partial(V \cap B_c), \mathbf{Z}) \oplus H^{2n}(H \cap B_c, \partial(H \cap B_c), \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. The map $lk_c : \pi_1(B_c - (V \cup H) \cap B_c) \to \mathbf{Z}$ corresponding to projection of $H_1(B_c - B_c \cap (V \cup H)$ onto $H^{2n}(B_c \cap V, \partial(B_c \cap V), \mathbf{Z}) = \mathbf{Z}$ geometrically is the linking number with V. Let U_c be the infinite cyclic cover of $B_c - (V \cup H) \cap B_c$ corresponding to the kernel of the homomorphism lk_c.

Definition 4.1. The order $P_c(t)$ of $H_n(U_c, \mathbf{Q})$ considered as the module over $\mathbf{Q}[t, t^{-1}]$ via the action induced by the deck transformations on U_c is called the polynomial of the singular point c.

Remark 4.2. U_c has a homotopy type of a finite dimensional complex and in particular $H_i(U_c, \mathbf{Q})$ is a torsion $\mathbf{Q}[t, t^{-1}]$-module for any i. So the above polynomial $P_c(t)$ is non zero. This follows from the following realization of the infinite cyclic cover U_c. Let ϕ_c be a holomorphic function in a neighbourhood N_c of c in $\mathbf{CP}^{n+1}$ such that $\phi_c = 0$ coincides with V in this neighbourhood. Let $V_c(s)$ be given by the equation $\phi_c = s$ in N_c. Then U_c is homotopy equivalent to $V_c(s) - H \cap V_c(s)$ for s sufficiently close to zero. Indeed the union of hypersurfaces $\phi_c = s$ ($s \le \epsilon$ and N_c is sufficiently small) is homeomorphic to the ball ([Mi]) and the function ϕ_c provides the locally trivial fibration of this ball over a punctured disk. Because the singularity of $V \cap H$ is isolated H will be transversal to all hypersurfaces $\phi_c = s$ (ϵ sufficiently small). Therefore $\phi_c(s)$ also provides the locally trivial

fibration of the complement in this ball to $V \cap H$. Hence $V_c(s) - V_c(s) \cap H$ is homotopy equivalant to U_c.

Theorem 4.3. If V and $V \cap H$ have only isolated singularities then the order P_V of $\pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}$ as the module over $\mathbf{Q}[t, t^{-1}]$ divides the product $\Pi_c P_c \cdot (t-1)^\kappa$ (for some $\kappa \in \mathbf{Z}$ of the polynomials P_c of all singularities of V including those in $\mathbf{C}^{n+1} = \mathbf{CP}^{n+1} - H$ as well as those at infinity. One can drop the factor $(t-1)^\kappa$ if one of the following conditions takes place:

a) $H^{n+1}(V, H \cap V, \mathbf{Q}) = 0$.

b) $\pi_n(\mathbf{CP}^{n+1} - (V \cap H) \otimes \mathbf{Q}$ is semisimple $\mathbf{Q}[t, t^{-1}]$-module.

Remark 4.4. One of the consequences of the theorem is that the order of $\pi_n(\mathbf{CP}^{n+1} - (V \cup H))$ is not zero. Hence this $\mathbf{Q}[t, t^{-1}]$-module is a torsion module.

Proof. Let $T(V)$ be a regular neighbourhood of V in $\mathbf{CP}^{n+1}$. First let us observe that $\pi_n(T(V) - ((T(V) \cap (H \cup V))$ surjects onto $\pi_n(\mathbf{CP}^{n+1} - (V \cap H))$. Indeed $T(V)$ contains a generic hypesurface W of the same degree as V which is transversal to $V \cup H$. According to the Lefschetz theorem $\pi_i(W - W \cap (V \cup H))$ maps by the map j_* induced by inclusion isomorphically to $\pi_i(\mathbf{CP}^{n+1} - (V \cup H))$ for $1 \leq i \leq n-1$ and surjects for $i = n$. Now our claim follows from the fact that j_* can be factored as

$$\pi_n(W - W \cap (V \cup H)) \to \pi_n(T(V) - T(V) \cap (V \cup H)) \to \pi_n(\mathbf{CP}^{n+1} - (V \cup H)). \quad (4.1)$$

.

Next let us consider a collection of non intersecting balls B_c about the singular points of V, $c \in Sing(V) \cup Sing_\infty(V)$. For a sufficiently small regular neighbourhood $T(V)$ of V the complement to the union of B_c ($c \in Sing(V) \cup Sing_\infty(V)$):

$$B_0 = T(V) - (T(V) \cap (V \cup H)) - \bigcup_c B_c \cap (T(V) - T(V) \cap (V \cup H)). \quad (4.2)$$

fiberes over $V - (H \cap V \bigcup_c (B_c \cap V))$ with the circle as the fibre. Such a fibre maps onto the generator of $H_1(\mathbf{C}^{n+1} - V, \mathbf{Z})$. Hence one has the surjection $lk_T : H_1(T(V) - T(V) \cap (V \cup H), \mathbf{Z}) \to H_1(\mathbf{C}^{n+1} - V, \mathbf{Z}) = \mathbf{Z}$ (the linking number, cf. Lemma 1.6). The kernel of lk_T defines the infinite cyclic cover U_T of $T(V) - T(V) \cap (V \cup H)$. For any c the map lk_c coincides with the composition of the map of the fundamental groups induced by embedding $B_c - B_c \cap V \to T(V) - T(V) \cap (V \cup H)$ and the map lk_T. A similar factorization takes place for the linking number homomorphism of $\pi_1(B_0)$, which defines the infinite cyclic cover U_0 of B_0. We obtain that

$$U_T = U_0 \bigcup_c U_c. \tag{4.3}$$

We claim that U_T is $n-1$-connected and in particular $\pi_n(T(V) - T(V) \cap (V \cup H) = H_n(U_T, \mathbf{Z})$. Indeed the fundamental group of U_T can be obtained from the fundamental group of U_0 by induction corresponding to adding U_c one by one using the van Kampen theorem (on π_1 of the union) . Each time the fundamental group is replaced by the quotient by the image of the fundamental group of the link of the corresponding singularity (in the case $c \in Sing_\infty(V)$ one rather should take the quotient by the image of the fundamental group of the complement to the intersection of the link of the singularity with the hyperplane at infinity inside this link). But the fundamental group of the affine portion of the smoothing of V which is transversal to H is calculated by the same procedure. Because the latter is simply connected we obtain that the fundamental group of U_T is trivial. On the other hand we have the following Mayer-Vietoris sequence:

$$\oplus_c H_i(U_0 \cap U_c) \to \oplus_c H_i(U_c) \oplus H_i(U_0) \to H_i(U_T) \to . \tag{4.4}$$

For $2 \le i \le n-1$ we have $H_i(U_c) = H_i(V_c(s)) = 0$. (Here, as in remark 4.2, $V_c(s)$ is a smoothing of the singularity c). This follows from the standard connectiviy of the Milnor fibre for finite singularities and for singularities at infinity follows from the latter and the exact sequence of the pair and $H_i(V_c(s) - V_c(s) \cap H, V_c(s)) = H_{i-2}(V_c(s) \cap H) = 0$ for

$2 \neq i \leq n$ (the first isomorphism is a consequence of excision and the Poincare duality). On the other hand if V' is a hypersurface in a pencil of hypersurfaces which contains V and has $V \cap H$ as the base locus then $H_i(V', V' \cap H, \cup_c V_c(s)) = H_i(V', V' \cap H) = 0$ ($c \in Sing(V) \cap Sing_{\infty}(V)$) for $i \leq n-1$ by Lefschetz theorem. Now excision of the union of small balls about all singular points of V shows that $H_i(U_0 \cap (\cup_c U_c)) = H_i(\cup_c \partial V_c(s), \partial V_c(s) \cap H) \rightarrow H_i(U_0) = H_i(V' - \cup V_c(s), V' - \cup V_c(s) \cap H)$ is surjective for $i \leq n-1$ and is isomorphism for $i \leq n-2$. Because $H_i(U_0 \cap U_c) = H_i(\partial V_c(s)) = 0$ for $0 < i < n-1$ we obtain that $H_i(U_T) = 0$ for $0 < i < n$.

For $i = n$ the sequence (4.4), being equivariant with respect to the action of $\mathbf{Z}$ by the deck transformations implies that if $Q = ord(Ker \oplus_c H_{n-1}(U_0 \cap U_c) \rightarrow \oplus_c H_{n-1}(U_c) \oplus H_{n-1}(U_0)) = ord(Ker \oplus_c (H_{n-1}(U_0 \cap U_c) \rightarrow H_{n-1}(U_0))$ and $R = ord(Ker \oplus_c H_n(U_0 \cap U_c) \rightarrow \oplus_c H_n(U_c) \oplus H_n(U_0))$ then $ord(H_n(U_T) = Q \cdot R$. The orders of $H_{n-1}(U_0 \cap U_c)$ and $H_n(U_0)$ are powers of $t-1$ because those are the cyclic covers of the (trivial) circle bundles. The order of $\oplus H_n(U_c)$ is equal to $\Pi_c P_c$ and hence R divides this product multiplied by a power of $t-1$. Therefore the order of $H_n(T(V) - T(V) \cap (V \cup H)$ divides $\Pi_c P_c$ multiplied by $(t-1)^{\kappa}$ for some κ. It follows from (4.1) that the same is true for order $P_V(t)$ of $\pi_n(\mathbf{CP}^{n+1} - (V \cup H)$. To conclude the proof we need to show that the order of zero of $P_V(t)$ at 1 does not exceed the sum of the orders of the zero at 1 of $P_c(t)$.

If one assumes a) above then according to the lemma 1.12 and 1.6 $P_V(1) \neq 0$ and the theorem follows. Moreover in the case b) the order of the zero of $P_V(t)$ at 1 is equal to the rank of $H^{n+1}(V, H \cap V, \mathbf{Q})$ as follows from the sequence (1.3). The order of the zero of $P_c(t)$ at 1 is greater or equal than the rank $H_{n-1}(L_c, \mathbf{Q})$ where L_c is the link of the singularity c (cf. [Mi]). Hence in the case b) the theorem follows from the inequality:

$$rkH^{n+1}(V, V \cap H, \mathbf{Q}) \leq \Sigma_c rkH_{n-1}(L_c, \mathbf{Q}) \tag{4.5}$$

To show this, note that $H^{n+1}(V, V \cap H, \mathbf{Q}) = H^{n+1}(V, V \cap H \cup S, \mathbf{Q}) = H^{n+1}(V - T(H) \cap V, (\partial T(H) \cap V) \cup T(S), \mathbf{Q})$ where S is the collection of the singular points of

V outside H, $T(S)$ is a small regular neighbourhood of this finite set in V and $T(H)$ is the regular neighbourhood of $V \cap H$ in V. The last cohomology group is dual to $H_{n-1}(V - V \cap H - S, \mathbf{Q})$ (use excision of $T(S)$). (4.5) will follow from the exact sequence of the pair and the vanishing of $H_{n-1}(V - S - V \cap H, \cup L_c, \mathbf{Q})$. This group is isomorphic to $H_{n-1}(\tilde{V} - H \cap V, \cup M_c, \mathbf{Q})$ where M_c is the Milnor fibre of the singularity c and $\tilde{V}$ is a generic hypersurface in the pencil of hypersurfaces containing V and having $V \cap H$ as the base locus. The vanishing of the last group is a consequence of the $n-1$ connectedness of the Milnor fibres M_c and the vanishing of $H_{n-1}(\tilde{V} - V \cap H)$ follows from the arguments just used and the exact sequence of the pair $(\tilde{V}, \tilde{V} - H \cap V)$.

Theorem 4.5. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ having only isolated singularities including infinity. Let H be the hyperplane at infinity. Let S_∞ be a sphere of a sufficiently large radius in $\mathbf{C}^{n+1} = \mathbf{CP}^{n+1} - H$ (or equivalently the boundary of a suffuciently small tubular neighbourhood of H in $\mathbf{CP}^{n+1}$). Let $U\infty$ be the infinite cyclic cover of $S_\infty - V \cap S_\infty$ corresponding to the kernel of the homomorphism: $\pi_1(S_\infty - V \cap S_\infty) \to \mathbf{Z}$ given by the linking number(cf. remark below). Let P_∞ be the order of $H_n(U_\infty, \mathbf{Q})$ as the $\mathbf{Q}[t, t^{-1}]$-module. Then P_V divides P_∞.

Remark 4.6. $V \cap S_\infty$ is a connected manifold if $n \geq 2$. If $n = 1$, the number of connected components of $V \cap S_\infty$ is "the number of places of the curve at infinity. By Alexander duality if $H_1(S_\infty - V \cap S_\infty, \mathbf{Z}) = H^{2n-1}(V \cap S_\infty, \mathbf{Z}) = \mathbf{Z}$ if $n \geq 2$ and for curves $H_1(S_\infty - S_\infty \cap V, \mathbf{Z})$ is a free abelian group of the rank equal to the number of places at infinity.

Remark 4.7. $H_n(U_\infty, \mathbf{Q})$ is a torsion module. Indeed let $L_c(V)$ (resp. $L_c(V \cap H)$) be the link of the singularity c of V (resp. $V \cap H$) in $\mathbf{CP}^{n+1}$ (resp. H). Let B_c be a polydisk in $\mathbf{CP}^{n+1}$ of the form $D_c^{2n} \times D_c^2$ about c such that the part of its boundary

$S^1_c \times D^{2n}_c \subset \partial T(H) = S_\infty$. Then

$$S_\infty - S_\infty \cap V = \partial T(H) - \partial T(H) \cap V = \bigcup_c (S^1_c \times D^{2n}_c - V) \cup U \tag{4.6}$$

where U fibres over $H - H \cap V$ with a circle as a fibre and hence the homology of the infinite cyclic cover of U is a torsion $\mathbf{Q}[t, t^{-1}]$-module (of the order which is a power of $t^d - 1$ where d is the degree of V because the circle about H is homologous to d multiplied by the generator of $H_1(\mathbf{CP}^{n+1} - (V \cup H)))$. $S^1_c \times D^{2n}_c - V = B_c - (V \cup H) \cap B_c$ and the homology groups of the infinite cyclic cover in question are $\mathbf{Q}[t, t^{-1}]$-torsion modules as follows from remark 4.2. Finally the intersection in the union (4.6) of U with $S^1_c \times D^{2n}_c - V \cap S^1_c \times D^{2n}$ fibres over $L_c(V \cap H)$ and hence the homology of its infinite cyclic cover is a torsion $\mathbf{Q}[t, t^{-1}]$ module as well. Hence the Mayer-Vietoris sequence yields the claim.

Proof of 4.5 . First note that $S_\infty \cap V$ is homotopy equivalent to $T(H) - T(H) \cap (V \cup H)$ where $T(H)$ is the tubular neighbourhood of H for which S_∞ is the boundary. If L is a generic hyperplane in $\mathbf{CP}^{n+1}$, which we will can assume belongs to $T(H)$, then once again by Lefschetz theorem we obtain that the composition:

$$\pi_n(L - L \cap (V \cup H) \rightarrow \pi_n(T(H) - T(H) \cap (V \cup H)) \rightarrow \pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \tag{4.7}$$

is surjective. Now the theorem follows from the multiplicativity of the order in exact sequences.

Corollary 4.8. If V is a hypersurface transversal to the hyperplane H at infinity then $\pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}$ is a semisimple $\mathbf{Q}[t, t^{-1}]$-module. Any root of the order P_V of the homotopy group is a root of 1 of degree d.

Proof. The surjectivity on the right homomorphism in (4.7) shows that the claim will follow from the semisimplicity of $\pi_n(T(H) - T(H) \cap (V \cup H) \otimes \mathbf{Q})$ as $\mathbf{Q}[t, t^{-1}]$-module. But $T(H) - T(H) \cap (V \cup H)$ is homotopy equivalent to the link of the singularity $\mathcal{V}_0$:

$z_1^d + ... + z_{n+1}^d = 0$ ($d = degV$) in $\mathbf{C}^{n+1}$ provided V is transversal to H. Indeed the projective closure of this hypersurface intersects H in a non-singular hypersurface which is isotopic to $V \cap H$ and this isotopy can be extended to a neighbourhood of $H \cap V$. The monodromy of $\mathcal{V}_0$ is semisimple (this is the case for any weighted homogeneous singularity because, as one can see from the explicit description of it (cf. [M]), this monodromy has a finite order). The last part in the statement of the corollary follows from the Milnor's formula for the characteristic polynomial of the monodromy of weighted homogeneous polynomials applied the singularity $\mathcal{V}_0$(cf. [M]).

Corollary 4.9. Let V be a hypersurface in $\mathbf{CP}^{n+1}$ given by equation $f = 0$. Assume that the singularities of V have codimension k in V. If V is transversal to the hyperplane H at infinity (as stratified space) then $\pi_k(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q})$ is isomorphic as $\mathbf{Q}[t, t^{-1}]$-module to $H_k(M_f, \mathbf{Q})$ where M_f is the Milnor fibre of the singularity at the origin in $\mathbf{C}^{n+2}$ of hypersurface $f(x_0, ..., x_{n+1}) = 0$ with the usual module structure given by the monodromy operator.

Proof. We can assume that the singularities of V are isolated because the general case can be reduced to this using Lefschetz theorems (cf. section 1). First notice that $\pi_n(\mathbf{CP}^{n+1} - V)$ is isomorphic to $H_n((\widetilde{\mathbf{CP}^{n+1} - V})_d, \mathbf{Z})$ where $(\widetilde{\mathbf{CP}^{n+1} - V})_d)$ is the $d = degV$-fold cyclic covering of $\mathbf{CP}^{n+1} - V$ because π_1 of the latter is $\mathbf{Z}/d\mathbf{Z}$. This d-fold covering is analytically equivalent to the affine hypersurface $f = 1$ which is diffeomorpfic to the Milnor fiber M_f. The deck transformation in this model of $(\widetilde{\mathbf{CP}^{n+1} - V})_d$ corresponds to the transformation induced by multiplication of each coordinate of $\mathbf{C}^{n+2}$ by a primitive root of unity of degree d. It is well known that this is also a description of the monodromy of a weighted homogeneous polynomial (cf. [M]). Finally according to lemma 1.13 $\pi_n(\mathbf{CP}^{n+1} - V) \otimes \mathbf{Q} = \pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}/(t^d - 1) \cdot \pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}$ which is isomorphic to $\pi_i(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q}$ because of the results which are contained in the corollary 4.8.

Remark 4.10 The corollary 4.9 is an extension to high dimensions of a result due to R.Randell [R] which gives a similar fact for Alexander polynomials. In the case of irreducible curves the divisibility theorem 4.3 gives a somewhat weaker result than the one in [L2] where it is shown that the Alexander polynomial divides the product the characteristic polynomials of all *branches* of the curve in all singular points.

5. Non trivial π_n

The purpose of this section is to prove two results which allow one to construct special classes of hypersurfaces with isolated singularities for which $\pi_n(\mathbf{C}^{n+1} - V) \otimes \mathbf{Q}$ is non-trivial. We start with the following lemma which may be of independent interest.

Lemma 5.1. Let $p(x_1, ..., x_{l+1})$ be a polynomial having a singularity of codimension k at the origin. Let $x_i = f_i(z_{i,1}, ..z_{i,n_i+1})$, $(i = 1, ..., l + 1)$ be a collection of polynomials all of which we also shall assume have at most isolated singularities at the origin. Suppose that $n_i \geq k + 1$ for every i for which f_i has singularity at the origin. Then the polynomial of $N = \sum_{i=1}^{l+1}(n_i + 1)$ variables $p(f) = p(f_1(z_{1,1}, ..., z_{1,n_1+1}), ..., f_{l+1}(z_{l+1,1}...z_{l+1,n_{l+1}+1}))$ has the singularity at the origin of codimension k. If M_p and $M_{p(f)}$ denote the Milnor fibres of the singularities of p and $p(f)$ respectively then $H_k(M_p, \mathbf{Z}) = H_k(M_{p(f)}, \mathbf{Z})$ as $\mathbf{Z}[t, t^{-1}]$ modules where the action of t in each case is given by the action of the monodromy operator.

Proof. First note that the use of induction allows one to reduce this lemma to the case when $f_i(z_{i,1}..., z_{i,n_i+1}) = z_{i,1}$ for $i \geq 2$, i.e. when the change of variables takes place only in one of x_i's. Let us select $\epsilon_1 > 0$ and small ball B_1 in $\mathbf{C}^{n_1+1}$ such that the intersection of the hypersurface $f(z_{1,1}, ..., z_{1,n_i+1}) = s$ with B_1, provided $0 < |s| \leq \epsilon_1$, is equivalent to the Milnor fibre of f_1. Let us consider a ball B_0 centered at the origin $\mathbf{C}^{l+1}$ of a radius less than ϵ_1. Let $\eta > 0$ be such that for $0 < |s| < \eta$ the portion of $p(x_1, ..., x_{l+1}) = s$

which belongs to B_0 is equivalent to the Milnor fibre of p. Let L be the intersection of M_η with the coordinate hyperplane $x_1 = 0$ in $\mathbf{C}^{l+1}$. Finally let us fix a polydisk $D \subset \mathbf{C}^{l+n_1+1}$ projection of which on subspace $x_2 = ...x_{l+1} = 0$ belongs to B_1 and such that intersection of it with $p(f) = s$ for $0 < |s| < \eta$ is equivalent to the Milnor fiber of $p(f)$. On a part D' of the polydisk D the formula:

$$x_1 = f_1(z_{1,1}, ..., z_{1,n_1+1}), x_i = z_{i,1}, (i = 2, ..., l+1) \tag{5.1}$$

defines a holomorphic map $F : D' \to B$. This map, when restricted on a Milnor fibre $M_{p(f)} \subset D'$ of $p(f)$ which is given by $p(f) = \eta$, takes $M_{p(f)}$ onto the Milnor fibre of p given by $p = \eta$. Let $\tilde{L}$ be the preimage of L: $F^{-1}(L)$. The restriction of F on $M_{p(f)} - \tilde{L}$ is a locally trivial fibration: $F : M_{p(f)} - \tilde{L} \to M_p - L$. The fibre of this fibration is equivalent to the Milnor fibre M_{f_1} of $f_1(z_{1,1}, ..., z_{1,n_1+1})$, This Milnor fibre is $(n_1 - 1)$ connected. The Leray spectral sequence: $E^2_{p,q} = H_p(M_p - L, H_q(M_{f_1}, \mathbf{Q}) \Rightarrow H_{p+q}(M_{p(f)} - \tilde{L}, \mathbf{Q})$ shows that the isomorphism $H_i(M_{p(f)} - \tilde{L}, \mathbf{Q}) = H_i(M_p - L, \mathbf{Q})$ will take place for i's which includes k and $k+1$. The following diagram which compares two exact sequences of pair:

$$\begin{array}{ccccccc}
H_{k+1}(M_p(f), M_p(f) - \tilde{L}, \mathbf{Q}) & \to & H_k(M_{p(f)} - \tilde{L}, \mathbf{Q}) & \to & H_k(M_{p(f)}, \mathbf{Q}) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
H_{k+1}(M_p, M_p - L, \mathbf{Q}) & \to & H_k(M_p - L, \mathbf{Q}) & \to & H_k(M_p, \mathbf{Q}) & \to
\end{array}$$

$$\begin{array}{cccccc}
\to & H_k(M_{p(f)}, M_{p(f)} - \tilde{L}, \mathbf{Q}) & \to & H_{k-1}(M_{p(f)} - \tilde{L}, \mathbf{Q}) & \to \\
 & \downarrow & & \downarrow & \\
\to & H_k(M_p, M_p - L, \mathbf{Q}) & \to & H_{k-1}(M_p - L, \mathbf{Q}) & \to
\end{array} \tag{5.2}$$

and the five lemma yields that the isomorphism of our lemma is a consequence of the isomorphism

$$H_i(M_{p(f)}, M_{p(f)} - \tilde{L}, \mathbf{Q}) \to H_i(M_p, M_p - L, \mathbf{Q}) for i = k, k+1 \tag{5.3}$$

Let $T(L)$ (resp. $T(\tilde{L})$) be the regular neighbourhoods of L (resp. $\tilde{L}$) in M_p (resp. $M_{p(f)}$) and $\partial T(L)$ (resp. $\partial T(\tilde{L})$) be the boundary of $T(L)$ (resp. $T(\tilde{L})$). Then using excision one obtains that $H_i(M_{p(f)}, M_{p(f)} - \tilde{L}, \mathbf{Q}) = H_i(T(\tilde{L}), \partial T(\tilde{L}), \mathbf{Q})$ and $H_i(M_p, M_p - L, \mathbf{Q}) =$

$H_i(T(L), \partial T(L), \mathbf{Q})$. But $H_i(T(\tilde{L}), \mathbf{Q}) = H_i(T(L), \mathbf{Q})$. Indeed $\tilde{L}$ fibres over L with contractible fibre (i.e. the part of $f_1 = 0$ inside B_1 which is the cone over the link of the singularity f_1) and $\partial T(\tilde{L})$ is a fibration over $\partial T(L)$ with $(n_1 - 1)$-connected fibre. Hence the last claimed isomorphism follows from the five lemma.

This lemma has the following corollary:

Theorem 5.2. For an integer k let $g_k(z_0, ..., z_{n+1})$ be a generic form of degree k. Let $V_{d_1,...,d_{n+1}}$ be a hypersurface of degree $D = d_1 \cdot d_2 \cdot ... \cdot d_{n+1}$ given by the equation:

$$g_{d_2 \cdot ... \cdot d_{n+1}}^{d_1} + g_{d_1 \cdot d_3 \cdot ... \cdot d_{n+1}}^{d_2} + ... + g_{d_1 \cdot d_2 \cdot ... \cdot d_n}^{d_{n+1}} = 0 \tag{5.4}$$

$V_{d_1,...,d_{n+1}}$ is a hypersurface in $\mathbf{CP}^{n+1}$ with D^n isolated singularities each of which is equivalent to the singularity at the origin of $q(x_1, ..., x_{n+1}) = x_1^{d_1} + ...x_{n+1}^{d_{n+1}}$. For a generic hyperplane $H \subset \mathbf{CP}^{n+1}$ one has the isomorphism:

$$\pi_n(\mathbf{CP}^{n+1} - (V \cup H)) \otimes \mathbf{Q} = H_n(M_q, \mathbf{Q}) \tag{5.5}$$

where M_q is the Milnor fiber of the singularity of q at the origin. This isomorphism is the isomorphism of $\mathbf{Q}[t, t^{-1}]$-modules where the module structure on the right is the one in which t acts as the monodromy operator.

Proof. The hypersurface $V_{d_1,...,d_{n+1}}$ is a section by a generic linear subspace of dimension $n+1$ of a hypesurface $\tilde{V}_{d_1,...d_{n+1}}$ in $\mathbf{CP}^{(n+2)(n+1)-1}$ given by:

$$G_{d_1,...,d_{n+1}} = \tilde{g}_{d_2 \cdot ... \cdot d_{n+1}}^{d_1}(z_{1,0}, z_{1,1}, ..., z_{1,n+1}) + ... + \tilde{g}_{d_1 \cdot ... \cdot d_n}^{d_{n+1}}(z_{n+1,0}, ..., z_{n+1,n+1}) = 0 \tag{5.6}$$

where $\tilde{g}_k$ are generic forms of disjoint set of variables. The hypersurface (5.6) has a singular locus containing the component of codimension $n+1$ in the ambient space which is given by $\tilde{g}_1 = ...\tilde{g}_{n+1} = 0$ (i.e. having codimension n inside the hypersurface) as well as possibly some components of larger codimensions. According to lemma 4.9 for generic hyperplane $\tilde{H}$

the module $\pi_i(\mathbf{CP}^{(n+2)(n+1)-1} - (\tilde{V}_{d_1,\ldots,d_{n+1}} \cup \tilde{H})) \otimes \mathbf{Q}$ is isomorphic to $H_n(M_{G_{d_1,\ldots,d_{n+1}}}, \mathbf{Q})$ with the usual $\mathbf{Q}[t,t^{-1}]$-module structure. The forms $\tilde{g}_{d_2\cdot\ldots\cdot d_{n+1}}, \ldots, \tilde{g}_{d_1\cdot\ldots\cdot d_n}$ have isolated singularities at the origins of corresponding $\mathbf{C}^{n+2}$ because of the genericity assumption. Hence, the preceeding lemma implies that $H_n(M_{G_{d_1,\ldots,d_{n+1}}}, \mathbf{Q})$ is isomorphic to $H_n(M_q, \mathbf{Q})$.

Proposition 5.3. Let $f_i = 0$ ($i = 1, 2$) be the equation of a hypersurface V_{f_i} of a degree d in $\mathbf{CP}^{n_i+1}$. Assume that the codimension of the singular locus of V_{f_i} is k_i. Then the hypersurface $V_{f_1+f_2}$ in $\mathbf{CP}^{n_1+n_2+3}$ given by $f_1 + f_2 = 0$ has the singular locus of codimension $k_1 + k_2 + 1$ and

$$\pi_{k_1+k_2+1}(\mathbf{CP}^{n_1+n_2+3} - V_{f_1+f_2}) \otimes \mathbf{Q} = (\pi_{k_1}(\mathbf{CP}^{n_1+1} - V_{f_1}) \otimes \mathbf{Q}) \otimes_{\mathbf{Q}} (\pi_{k_2}(\mathbf{CP}^{n_2+1} - V_{f_2}) \otimes \mathbf{Q}) \tag{5.7}$$

.

Proof. This is an immediate consequence of (4.9) and the Thom-Sebastiani theorem.

Examples 5.4. 1. Let $f(x_0, x_1, x_2) = 0$ be an equation of a curve C of degree 6 which has 6 cusps on a conic. The homology of the infinite cyclic cover of $\mathbf{CP}^2 - (C \cup L)$ for generic line is $\mathbf{Q}[t, t^{-1}]/(t^2 - t + 1)$. (cf. [L]. Recall that $\pi_1(\mathbf{CP}^2 - (C \cup L))$ is the braid group on 3 strings, i.e the group of the trefoil knot, and hence the homology in question is the Alexander module of the trefoil. Let $g(y_0, y_1, y_2) = 0$ be an equation of another sextic with six cusps also on a conic. According to proposition (5.3) (in which the homotopy groups in case of curves are replaced by the Alexander modules) the generic section by $\mathbf{CP}^4$ of the hypersurface in $\mathbf{CP}^6$ given by

$$f(x_0, x_1, x_2) + g(y_0, y_1, y_2,) = 0 \tag{5.8}$$

is a threefold W which has isolated singularities (the number of which is 6^3) and the order of the homotopy group $\pi_3 \otimes \mathbf{Q}$ of the complement is $(t^2 - t + 1)$. If one takes as f in (5.8)

the equation of a sextic with nine cusps which is dual to a non singular cubic and uses the fact that the Alexander module for its complement is $(\mathbf{Q}[t,t^{-1}]/(t^2-t+1))^{\oplus 3}$. This is a consequence of the calculation of the fundamental group of the complement to such a curve due to O.Zariski. He found that the fundamental group of this curve is the kernel of the canonical map of the braid group of the torus $S^1 \times S^1$ onto $H_1(S^1 \times S^1, \mathbf{Z})$ (cf. [Z]) Combining this with the calculation which uses the Fox calculus and the presentation for the braid group of the torus one arrives to the Alexander module as above. Therefore one obtains the threefold W with $\pi_3(\mathbf{CP}^4 - W) \otimes \mathbf{Q}$ being the same as the Alexander module just mentioned. Iteration of this construction obtained by replacing in (5.8) g by the equation of an n-dimensional hypersurface with isolated singularities and non vanishing homotopy group $\pi_n \otimes \mathbf{Q}$ of the complement gives examples of hypersurfaces of degree 6 and dimension n for which $\pi_n \otimes \mathbf{Q}$ has arbitrary large rank for sufficiently large n.

2. Let us consider the equation (5.8) in which $f(x_0, x_1, x_2)$ is the form giving the equation of a sextic with six cusps not on conic. The fundamental group of the complement to such a curve is abelian (cf. [Z]) and therefore the homology of the universal cyclic cover of the complement is trivial. Hence in this case the construction of example 1 yields a threefold W' which has $\pi_3(\mathbf{CP}^4 - W') \otimes \mathbf{Q} = 0$ but which has the same degree (i.e. 6) and the same number of singularities (i.e. 36) of the same type as W (i.e. locally given by $x^2 + y^2 + u^3 + v^3 = 0$).

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